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The η-invariant of mapping tori with finite monodromies

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Abstract

The η-invariant of Riemannian 3-manifolds is defined by means of the spectrum of a certain elliptic operator. In this paper, we give a geometric interpretation of the deviation from the multiplicativity of the η-invariant for finite coverings. We then apply it to mapping tori with finite monodromies, and obtain a simple formula of the η-invariant for it.

Keywords: η-invariant; Finite covering; Canonical 2-framing; Mapping class group

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0. Introduction

The definition of the η-invariant was originally given by Atiyah, Patodi and Singer [3] in terms of the spectrum of a self-adjoint elliptic operator. The η-invariant in which we are interested here is that of the signature operator on 3-dimensional Riemannian manifolds. The main result of [3] shows that this invariant is equal to the integral of the first Pontrjagin form minus the signature of 4-manifold which allows us to compute it without using analytic tools. However it should be emphasized that this is not a topological invariant, but a spectral invariant.

In general, the η-invariant of the total space of a finite covering is not the multiple of that of the base space by the factor of the covering degree, and in fact the deviation of the multiplicativity is calculated by using the G-signature theorem (see [1,5]). In our point of view, we will regard this deviation as the difference between (Atiyah's) canonical 2-framings [2] of the total space and the base space. If we consider the case where the total space admits an orientation reversing isometry, then by definition its η-invariant vanishes. Therefore it follows that calculating the η-invariant of the base space and the difference

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between the 2-framings are equivalent each other. In particular, for a mapping torus M_φ corresponding to an element φ of the mapping class group $\mathcal{M}_g$ of an oriented closed surface Σ_g of genus g, we can describe the difference by using the 2-cocycle $c(\varphi,\psi)$ which corresponds to a certain central extension of $\mathcal{M}_g$ (see Section 3 for details). The main result of this paper is the following.

Main Theorem. *Let $\widetilde{\varphi}$ be an orientation preserving diffeomorphism of Σ_g of finite order m and $\varphi \in \mathcal{M}_g$ be its mapping class. Then the η-invariant of the mapping torus $M_{\widetilde{\varphi}^n}$ $(1 \leqslant n \leqslant m)$ is given by*

$$\eta(M_{\widetilde{\varphi}^n}) = \frac{1}{3}\left\{\sum_{k=1}^{n-1} c(\varphi,\varphi^k) - \frac{n}{m}\sum_{k=1}^{m-1} c(\varphi,\varphi^k)\right\}.$$

Here we endow $M_{\widetilde{\varphi}^n}$ with the metric which is induced from the product metric of any metric for which $\widetilde{\varphi}$ acts as an isometry on Σ_g and the standard metric on S^1 via the projection $\Sigma_g \times S^1 \to M_{\widetilde{\varphi}^n}$.

Now we describe the contents of this paper. In Section 1 we recall the definitions of the η-invariant and the canonical 2-framing according to [2] and [3]. In Section 2 we give a formula of the η-invariant for finite coverings of Riemannian 3-manifolds. By this formula, we see the equivalence mentioned above. In Section 3 we compute the difference between the 2-framings for mapping tori.

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1. Definitions of η-invariant and canonical 2-framing

As shown in [3], the η-invariant of 3-manifolds measures the extent to which the Hirzebruch signature formula fails for a non-closed 4-dimensional Riemannian manifold whose metric is a product near its boundary.

Theorem 1.1 [3]. *Let W be a 4-dimensional compact oriented Riemannian manifold with boundary M and assume that, near M, it is isometric to a product. Then*

$$\eta(M) = \frac{1}{3}\int_W p_1 - \operatorname{Sign} W,$$

where $\operatorname{Sign} W$ is the signature of the nondegenerate quadratic form defined by the cup product on the image of $H^2(W,M)$ in $H^2(W)$ and p_1 is the first Pontrjagin form of the Riemannian metric.

In the following, we will consider the above description as a definition of $\eta(M)$. For example, in the case of $M = S^3$, the η-invariant of S^3 with respect to the standard metric vanishes, because S^3 has an orientation reversing isometry.

Example 1.2 [4]. The η-invariant of the 3-dimensional lens space $L(p,q)$ is given by

$$\eta\big(L(p,q)\big) = -\frac{1}{p}\sum_{k=1}^{p-1}\cot\left(\frac{k}{p}\pi\right)\cot\left(\frac{kq}{p}\pi\right) = -4s(q,p).$$

Here (p,q) is a coprime pair of integers and $s(q,p)$ is the Dedekind sum (see [8]).

Next we review the definition of the 2-framings briefly. See Atiyah [2] for details. Let M be a closed oriented 3-manifold. We call a trivialization of twice the tangent bundle of M, denoted by $2TM = TM \oplus TM$, as a Spin(6)-bundle a 2-framing of M. Atiyah has shown that there is a canonical choice for these 2-framings. It is characterized by the following property: If W is an oriented 4-manifold such that $\partial W = M$, then the signature of W is

$$\operatorname{Sign} W = \tfrac{1}{6}p_1(2TW, t_M),$$

where the relative Pontrjagin number is computed by using the 2-framing t_M on the boundary. We call t_M the canonical 2-framing of M.

Remark. The Hirzebruch formula continues to hold when $M = \partial W$ is not connected, provided each component of M is given its canonical 2-framing.

2. A formula for finite coverings

In this section, we study the η-invariant for finite coverings. Let W and M be as in Theorem 1.1 above. Namely, suppose that W is isometric to a product $M \times [0,1]$ near M, where $M = M \times 0$. We set $W_0 = W - M \times [0,1)$. Moreover let $F(W)$ be the SO(4) oriented frame bundle of W.

We calculate the integral of the first Pontrjagin form in Theorem 1.1 by using a connection on $2TW$ which is defined as follows: Let $F(M)$ be the SO(3) oriented frame bundle of M and ω_g be the Levi–Civita connection on $F(M)$. Let ω_{α_0} be the (flat) connection on $F(M)$ defined by an orthonormal framing α_0. Choose a smooth monotonic function μ on $[0,1]$ satisfying $0 \leqslant \mu \leqslant 1$, $\mu([0,\frac{1}{3}]) = 0$ and $\mu([\frac{2}{3},1]) = 1$. For $t \in [0,1]$, let ω_t be the connection on $F(M)$ defined by

$$\omega_t = \big(1-\mu(t)\big)\omega_g + \mu(t)\omega_{\alpha_0},$$

where $+$ is taken in the convex linear space of all the smooth connections on $F(M)$. Then $\omega_0 = \omega_g$ and $\omega_1 = \omega_{\alpha_0}$. Let ω be the connection on $F(M \times [0,1])$ such that $\omega = \omega_t$ on $F(M \times t)$ and ω is trivial in the direction of t. Extend ω to a smooth connection on $F(W)$ in an arbitrary way on $F(W_0)$ and we get a smooth connection on $F(W)$. We denote it by ω again.

For twice the tangent bundle of W, we define the desired connection on $2TW$ by $2\omega = \omega \oplus \omega$. Then direct calculation shows that

$$\int_W p_1 = \frac{1}{2}\int_M \alpha^*\mathrm{CS} + \frac{1}{2}p_1(2TW, \alpha),$$

where $p_1(2TW, \alpha)$ is the relative Pontrjagin number with respect to the 2-framing α which is induced by α_0 and CS is the Chern–Simons form [6] of the connection on $2TW$ induced by the metric. Here we think of the 2-framing α as a section.

Now Sign W is an integer by definition and $\frac{1}{2}p_1(2TW, \alpha)$ is also an integer, so that we can evalute the difference from the canonical 2-framing t_M by an integer value.

Definition 2.1. For a 2-framing α on 3-manifold M, we define the difference degree $d(\alpha; t_M)$ by

$$d(\alpha; t_M) = \tfrac{1}{2}p_1(2TW, \alpha) - 3 \operatorname{Sign} W \in \mathbb{Z}.$$

Remark. $d(\alpha; t_M)$ does not depend on the choice of W.

By using the difference degree, we obtain

$$\eta(M) = \frac{1}{6}\int_M \alpha^*\mathrm{CS} + \frac{1}{3}d(\alpha; t_M).$$

In particular if $\alpha = t_M$,

$$\eta(M) = \frac{1}{6}\int_M t_M{}^*\mathrm{CS}.$$

We now consider the next situation. Let $\pi : \widetilde{M} \to M$ be an n-fold Riemannian covering

$$\begin{tikzcd}
2T\widetilde{M} \arrow[r, "\tilde{\pi}"] & 2TM \\
\widetilde{M} \arrow[u, "t_{\widetilde{M}}"', leftarrow] \arrow[r, "\pi"] \arrow[u, bend left=60] & M \arrow[u, "t_M", leftarrow] \arrow[u, bend right=60]
\end{tikzcd}$$

where $\tilde{\pi}$ is the lift of the projection π. For $M = \widetilde{M}$ and $\alpha = \pi^* t_M$ (the lift of t_M), we have

$$\eta(\widetilde{M}) = \frac{1}{6}\int_{\widetilde{M}} (\pi^* t_M)^*\widetilde{\mathrm{CS}} + \frac{1}{3}d(\pi^* t_M; t_{\widetilde{M}}),$$

where $\widetilde{\mathrm{CS}}$ is the Chern–Simons form on $2T\widetilde{W}$ ($\partial\widetilde{W} = \widetilde{M}$). Then we can compute the first term as follows:

$$\begin{aligned}
\frac{1}{6}\int_{\widetilde{M}} (\pi^* t_M)^*\widetilde{\mathrm{CS}} &= \frac{1}{6}\int_{\widetilde{M}} (\pi^* t_M)^*(\tilde{\pi}^*\mathrm{CS}) = \frac{1}{6}\int_{\widetilde{M}} \pi^*(t_M{}^*\mathrm{CS}) \\
&= n \cdot \frac{1}{6}\int_M t_M{}^*\mathrm{CS} = n\eta(M).
\end{aligned}$$

Therefore we get the next simple formulation.

Proposition 2.2. *Let $\pi : \widetilde{M} \to M$ be an n-fold Riemannian covering. Then the deviation from the multiplicativity of the η-invariant is described by the difference between the canonical 2-framings:*

$$\eta(\widetilde{M}) = n\eta(M) + \tfrac{1}{3}d(\pi^* t_M; t_{\widetilde{M}}).$$

In particular, if $\widetilde{M}$ admits an orientation reversing isometry, the above proposition implies that calculating the η-invariant of M and the difference degree are equivalent.

Corollary 2.3. *For the p-fold covering $\pi : S^3 \to L(p,q)$ with respect to the standard metric, $d(\pi^* t_{L(p,q)}; t_{S^3}) = 0$ if and only if $q^2 \equiv -1 \pmod p$. In other words, the following two conditions are equivalent:*

(i) *The lift of $t_{L(p,q)}$, $\pi^* t_{L(p,q)}$ coincides exactly with t_{S^3} on S^3. That is, the canonical 2-framing of S^3 is invariant under a cyclic group action of order p.*

(ii) *$L(p,q)$ has an orientation reversing self-isometry.*

Proof. It is well known that $L(p,q)$ has an orientation reversing isometry if and only if $q^2 \equiv -1 \pmod p$. Hence 'only if' part is trivial. Conversely, suppose $d = 0$. Then by using the reciprocity law of the Dedekind sum (see [8]), we have $0 = 12pqs(p,q) = p^2 + q^2 + 1 - 3pq$. It follows that $q^2 \equiv -1 \pmod p$ and this completes the proof. □

3. Difference degree of mapping tori

In this section, we compute the difference degree of mapping tori. As a result, we can obtain the main theorem of this paper. See Atiyah [1,2] and Meyer [7].

Let Σ_g be an oriented closed surface of genus g ($g \geqslant 1$) and $\mathcal{M}_g$ be its mapping class group. Namely, it is the group of all isotopy classes of orientation preserving diffeomorphisms of Σ_g. For $\varphi \in \mathcal{M}_g$, let M_φ be a mapping torus with monodromy φ. More precise, $M_\varphi = \Sigma_g \times I / \sim$ where we identify $(x,0)$ with $(\varphi(x),1)$ $(x \in \Sigma_g)$.

Now we recall the definition of a central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widehat{\mathcal{M}}_g \longrightarrow \mathcal{M}_g \longrightarrow 1.$$

As a set, $\widehat{\mathcal{M}}_g$ is all pairs of (φ, α), where $\varphi \in \mathcal{M}_g$ and α is a 2-framing on M_φ. Let $W_{(\varphi,\psi)}$ $(\varphi, \psi \in \mathcal{M}_g)$ be a Σ_g-bundle over 2-sphere with three holes (or pair of pants), whose boundaries are $M_\varphi \cup M_\psi \cup (-M_{\varphi\psi})$. Then the group law of $\widehat{\mathcal{M}}_g$ is defined by $(\varphi,\alpha)(\psi,\beta) = (\varphi\psi,\gamma)$, where γ is the 2-framing on $M_{\varphi\psi}$ such that the relative p_1 of $W_{(\varphi,\psi)}$ with respect to the trivialization $\alpha + \beta - \gamma$ on $\partial W_{(\varphi,\psi)}$ vanishes.

Let us consider the cohomology class of this extension. We define the canonical section $s : \mathcal{M}_g \to \widehat{\mathcal{M}}_g$ by $s(\varphi) = (\varphi, t_{M_\varphi})$ $(\varphi \in \mathcal{M}_g)$. This induces the associated 2-cocycle of the above extension

$$c(\varphi,\psi) = s(\varphi)s(\psi)\{s(\varphi\psi)\}^{-1}.$$

This is the integer difference between the two 2-framings $s(\varphi)s(\psi)$ and $s(\varphi\psi)$ on $M_{\varphi\psi}$ and we call it the canonical 2-cocycle.

According to the definition of multiplication in the group $\widehat{\mathcal{M}_g}$, the relative p_1 of the 4-manifold $W_{(\varphi,\psi)}$ with respect to the trivialization $s(\varphi)+s(\psi)-s(\varphi)s(\psi)$ on its boundary vanishes. Thus we have

$$\begin{aligned}
c(\varphi,\psi) &= s(\varphi)s(\psi)\{s(\varphi\psi)\}^{-1} \\
&= -\tfrac{1}{2}p_1\bigl(2T(M_{\varphi\psi}\times I), -s(\varphi)s(\psi)+s(\varphi\psi)\bigr) \\
&= \tfrac{1}{2}p_1\bigl(2TW_{(\varphi,\psi)}, s(\varphi)+s(\psi)-s(\varphi\psi)\bigr),
\end{aligned}$$

where we trivialize the boundary of $M_{\varphi\psi}\times I$ by $-s(\varphi)s(\psi)$ on $-M_{\varphi\psi}\times 0$ and $s(\varphi\psi)$ on $M_{\varphi\psi}\times 1$.

Next we examine a connection between the canonical 2-cocycle and the difference degree. Now let us recall the definition of the difference degree. Let W be a 4-manifold such that $\partial W = \widetilde{M}$ and $\pi : \widetilde{M} \to M$ be a finite covering. Then

$$\begin{aligned}
d(\pi^* t_M; t_{\widetilde{M}}) &= \tfrac{1}{2}p_1(2TW, \pi^* t_M) - 3\,\mathrm{Sign}\, W \\
&= \tfrac{1}{2}p_1(2TW, \pi^* t_M) - \tfrac{1}{2}p_1\bigl(2TW, t_{\widetilde{M}}\bigr) \\
&= -\tfrac{1}{2}p_1\bigl(2T(\widetilde{M}\times I), -\pi^* t_M + t_{\widetilde{M}}\bigr).
\end{aligned}$$

Applying this to a mapping torus M_φ, we can compute its difference degree. From now on we simply denote the canonical 2-framing of a mapping torus M_φ by t_φ.

Proposition 3.1. *Let $f_n : M_{\varphi^n} \to M_\varphi$ be an n-fold covering. Then the difference degree of the 2-framing ${f_n}^* t_\varphi$ is given by*

$$d({f_n}^* t_\varphi; t_{\varphi^n}) = \sum_{k=1}^{n-1} c(\varphi, \varphi^k).$$

Remark. This formula holds for any mapping classes (not necessarily finite orders).

Proof. First, we consider a 2-fold covering $f_2 : M_{\varphi^2} \to M_\varphi$. For the 4-manifold $W_{(\varphi,\varphi)}$, we trivialize its boundary $\partial W_{(\varphi,\varphi)} = M_\varphi \cup M_\varphi \cup (-M_{\varphi^2})$ by $t_\varphi + t_\varphi - {f_2}^* t_\varphi$. Moreover for $M_{\varphi^2}\times I$, we trivialize its boundary by $-{f_2}^* t_\varphi$ on $-M_{\varphi^2}\times 0$ and t_{φ^2} on $M_{\varphi^2}\times 1$, where we regard $-M_{\varphi^2}\times 0$ as the original boundary component of $W_{(\varphi,\varphi)}$. Then we have

$$\begin{aligned}
d({f_2}^* t_\varphi; t_{\varphi^2}) &= -\tfrac{1}{2}p_1\bigl(2T(M_{\varphi^2}\times I), -{f_2}^* t_\varphi + t_{\varphi^2}\bigr) \\
&= -\tfrac{1}{2}p_1\bigl(2TW_{(\varphi,\varphi)}, t_\varphi + t_\varphi - {f_2}^* t_\varphi\bigr) \\
&\quad + \tfrac{1}{2}p_1\bigl(2TW_{(\varphi,\varphi)}, t_\varphi + t_\varphi - t_{\varphi^2}\bigr).
\end{aligned}$$

Since the 2-framing ${f_2}^* t_\varphi$ is invariant under a cyclic group action of order 2, we see that ${f_2}^* t_\varphi$ coincides with twice of t_φ. Accordingly the first term of the above formula vanishes. On the other hand, it is clear that the second term is equal to the canonical 2-

cocycle $c(\varphi,\varphi)$, since we trivialize the boundary $M_\varphi \cup M_\varphi \cup (-M_{\varphi^2})$ by their canonical 2-framings respectively. Hence for a 2-fold covering of the mapping torus, it follows that

$$d({f_2}^* t_\varphi; t_{\varphi^2}) = c(\varphi,\varphi).$$

Next, we consider a 3-fold covering $f_3 : M_{\varphi^3} \to M_\varphi$. Then it is easy to see that the difference degree $d({f_3}^* t_\varphi; t_{\varphi^3})$ coincides with

$$-\tfrac{1}{2}p_1\big(2TW_{(\varphi,\varphi^2)}, t_\varphi + t_{\varphi^2} - {f_3}^* t_\varphi\big) + \tfrac{1}{2}p_1\big(2TW_{(\varphi,\varphi^2)}, t_\varphi + t_{\varphi^2} - t_{\varphi^3}\big).$$

Now let $W_{(\varphi,\varphi,\varphi)}$ be a Σ_g-bundle over a 2-sphere with four holes, whose boundaries are $M_\varphi \cup M_\varphi \cup M_\varphi \cup (-M_{\varphi^3})$. If we glue $W_{(\varphi,\varphi)}$ to $W_{(\varphi,\varphi^2)}$ along the boundary component M_{φ^2}, the trivialization on M_{φ^2} vanishes. Consequently the first term of the above formula is equal to

$$-\tfrac{1}{2}p_1\big(2TW_{(\varphi,\varphi,\varphi)}, t_\varphi + t_\varphi + t_\varphi - {f_3}^* t_\varphi\big) + \tfrac{1}{2}p_1\big(2TW_{(\varphi,\varphi)}, t_\varphi + t_\varphi - t_{\varphi^2}\big),$$

by the additivity of the relative p_1. From the same reason for the case of $n = 2$, the relative p_1 of $W_{(\varphi,\varphi,\varphi)}$ is zero. The other terms can be described by the canonical 2-cocycles, so that we get

$$d({f_3}^* t_\varphi; t_{\varphi^3}) = c(\varphi,\varphi) + c(\varphi,\varphi^2).$$

For the general case, we can repeat the above argument. The assertion follows and this completes the proof of Proposition 3.1. □

Combining Propositions 2.2 and 3.1, and applying it to an orientation preserving diffeomorphism of Σ_g of finite order, we can obtaion the main theorem.

Example 3.2. Hyperelliptic involution $\varphi : \Sigma_g \to \Sigma_g$ (namely $\varphi^2 = \mathrm{id}$).

Consider the 2-fold covering $M_{\varphi^2} = \Sigma_g \times S^1 \to M_\varphi$. Then φ has $2g + 2$ isolated fixed points on Σ_g, and the rotation angle at each point is π. Thus the contribution from the fixed points set vanishes. Hence $\eta(M_\varphi) = 0$ by the G-signature theorem (see [1]).

On the other hand, we see the above fact immediately by our main theorem. More generally, it follows that the η-invariant of a mapping torus corresponding to any involution vanishes. (In other words, the canonical 2-framing of the product 3-manifold $\Sigma_g \times S^1$ is invariant under a $\mathbb{Z}_2$-action determined by any involution of Σ_g.)

Example 3.3. The case of torus bundles (that is $g = 1$).

It is known that the canonical 2-cocycle $c(\varphi,\psi)$ is equal to minus thrice of Meyer's signature cocycle $\tau(\varphi,\psi)$ (see [1,2] and [7], in particular note that $\tau(\varphi,\psi)$ represents the minus signature of $W_{(\varphi,\psi)}$). Accordingly we have

$$d({f_n}^* t_A; t_{A^n}) = -3\sum_{k=1}^{n-1} \tau(A, A^k) \quad (A \in \mathrm{SL}(2,\mathbb{Z})).$$

Using this, we can calculate the η-invariant for the torus bundles determined by the elements of $\mathrm{SL}(2,\mathbb{Z})$ of finite orders. Such elements are elliptic (namely, $|\mathrm{tr}\,A| < 2$).

(i) $\mathrm{tr}A = 1$. In this case, A has order 6. For example, we can take

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

$$\eta(M_A) = -\eta(M_{A^5}) = -\tfrac{4}{3}, \qquad \eta(M_{A^2}) = -\eta(M_{A^4}) = -\tfrac{2}{3},$$

$$\eta(M_{A^3}) = \eta(M_{A^6}) = 0.$$

(ii) $\mathrm{tr}A = 0$. In this case, A has order 4. For example,

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$$\eta(M_A) = -\eta(M_{A^3}) = -1, \qquad \eta(M_{A^2}) = \eta(M_{A^4}) = 0.$$

(iii) $\mathrm{tr}A = -1$. In this case, A has order 3. For example,

$$A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

$$\eta(M_A) = -\eta(M_{A^2}) = -\tfrac{2}{3}, \qquad \eta(M_{A^3}) = 0.$$

These values coincide with the results obtained by Atiyah in [1].

Finally we examine the case of a higher genus. For simplicity we calculate only for Σ_2.

Example 3.4. The case of Σ_2-bundles (orders 5 and 10).

Let φ (respectively ψ) be a diffeomorphism on Σ_2, defined in [9, p. 258], of order 5 (respectively 10). We choose a symplectic basis of $H = H_1(\Sigma_2,\mathbb{Z})$ as usual and fix it. Representing φ and ψ as the elements of the Siegel modular group $\mathrm{Sp}(4,\mathbb{Z})$ by investigating the actions on H, we have the following matrices (we use the same letters):

$$\varphi = \begin{pmatrix} -1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ -2 & 2 & 2 & 1 \\ 1 & -2 & -2 & -2 \end{pmatrix}, \qquad \psi = -\varphi.$$

Therefore we can compute Meyer's cocycle and obtain the η-invariant of mapping tori corresponding to φ and ψ:

$$\eta(M_{\varphi^n}) = \frac{8n}{5} - 4\left[\frac{n}{2}\right] \quad (1 \leqslant n \leqslant 5)$$

$$\eta(M_{\psi^n}) = -\frac{12n}{5} + 4\left[\frac{2n}{3}\right] \quad (1 \leqslant n \leqslant 10).$$

Here $[x]$ is the Gaussian symbol (that is the largest integer less than or equal to x).

In principle, similar computations can be made for cases of higher genera, although they will be more complicated.

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